PERGAMON

International Journal of Solids and Structures 37 (2000) 3457–3477

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A novel approach to the solution of the tensor equation $\mathbf{AX}+\mathbf{XA}=\mathbf{H}$

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Received 26 February 1998; in revised form 2 February 1999

Abstract

A systematic approach to the solution of the tensor equation $\mathbf{AX}+\mathbf{XA}=\mathbf{H}$, where $\mathbf{A}$ is symmetric, is presented. It is based upon the reformulation of the original equation in the form $\mathbb{A}\mathbf{X}=\mathbf{H}$ where $\mathbb{A}=\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}$ is the fourth-order tensor obtained from the square tensor product of the second-order tensors $\mathbf{A}$ and $\mathbf{1}$. It is shown that the solution $\mathbf{X}$, which is known to be an isotropic function of $\mathbf{A}$ and $\mathbf{H}$, can be effectively obtained either by providing explicit formulas for $\mathbb{A}^{-1}$ or by reconverting to the format $\mathbb{A}\mathbf{X}=\mathbf{H}$ the well-known representation formulas for tensor-valued isotropic functions. The final form of the solution can thus be established a priori by suitably choosing a set of independent generators for $\mathbb{A}^{-1}$. The coefficients of the expansion of $\mathbb{A}^{-1}$ with respect to the assigned generators are then obtained by means of basic composition rules for square tensor products. In this way it is possible to provide new expressions of the solution as well as to derive the existing ones in a simpler way. Both three-dimensional and two-dimensional cases are addressed in detail.

1. Introduction

Several problems in mechanics of solids require the solution of the tensor equation

$$\mathbf{AX}+\mathbf{XA}=\mathbf{H} \tag{1}$$

in the unknown $\mathbf{X}$. We denote by $\mathbf{A}$, $\mathbf{X}$ and $\mathbf{H}$ second-order tensors on a two- or three-dimensional inner product space V over the real numbers and we assume $\mathbf{A}$ to be symmetric.

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PII: S0020-7683(99)00053-0

It is immediate to verify that $\mathbf{X}$ is symmetric (skew) if and only if $\mathbf{H}$ is symmetric (skew). Further, as first observed by Sidoroff (1978), $\mathbf{X}$ is an isotropic function of $\mathbf{A}$ and $\mathbf{H}$, linear in $\mathbf{H}$.

A comprehensive review of applications of Eq. (1) in continuum mechanics and other branches of physics and engineering can be found, e.g. in Scheidler (1994). For instance, denoting by $\mathbf{U}$ and $\mathbf{V}$ the right and left stretch tensors (Gurtin, 1981), their material time derivatives $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ can be obtained in turn by solving the tensor equations

$$\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} = \dot{\mathbf{C}} \quad \text{and} \quad \mathbf{V}\dot{\mathbf{V}} + \dot{\mathbf{V}}\mathbf{V} = \dot{\mathbf{B}}$$

where

$$\mathbf{C} = \mathbf{F}^{\mathrm{t}}\mathbf{F} = \mathbf{U}^2 \quad \text{and} \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{t}} = \mathbf{V}^2$$

are the right and left Cauchy–Green tensors, $\mathbf{F}$ is the deformation gradient and $(\cdot)^{\mathrm{t}}$ denotes transposition of the argument $(\cdot)$.

For the applications in continuum mechanics the tensor $\mathbf{A}$ in Eq. (1) is in most cases symmetric and positive definite even if the results shown in the paper will hold true under the less restrictive hypotheses of a symmetric nonsingular $\mathbf{A}$ which does not have two eigenvalues equal and opposite in sign. Actually, as proved in Section 3, these conditions ensure that Eq. (1) has a unique solution for every $\mathbf{H}$.

The solution of tensor equations more general than (1) has been addressed by Smith (1966), Jameson (1968), Muller (1970) and Kucera (1974). They however established quite complicated results that make inefficient their applicability to the case at hand.

Simpler results were obtained by Sidoroff (1978) who was the first one to find a direct solution of Eq. (1), i.e. a solution expressed solely in terms of $\mathbf{A}$ and $\mathbf{H}$. Additional expressions of the solution of Eq. (1) were later found by Dienes (1979) and Guo (1984) for $\mathbf{H}$ skew, by Hoger and Carlson (1984) for a generic $\mathbf{H}$ and by Mehrabadi and Nemat-Nasser (1987) who addressed the case of a skew-symmetric right-hand side of Eq. (1) having a slightly different form.

However the previous authors did not provide a general methodology for solving the tensor Eq. (1). This was mainly due to the fact that the solution of Eq. (1) was only an intermediate result to be used in the subsequent developments of the specific problem addressed in their paper.

Further, in some cases, the solution was derived by following an indirect approach. Actually, the original coordinate system, in which $\mathbf{A}$ and $\mathbf{H}$ were assigned, was transformed to a new system having axes coincident with the principal directions of the tensor $\mathbf{A}$. As a consequence, the solution obtained in the new coordinate system had to be converted back to the original one.

For instance, the solution strategy exploited by Sidoroff (1978) and Guo (1984), required the use of the axial vector associated with a skew $\mathbf{H}$. Only subsequently the solution thus obtained was expressed as function of $\mathbf{H}$.

Also the path followed by Mehrabadi and Nemat-Nasser (1987) was partly indirect. They actually obtained the solution of Eq. (1) by repeated use of the Cayley–Hamilton theorem which somehow prompted the need of a more direct approach.

The only remarkable exception among the previous papers is the contribution by Hoger and Carlson (1984) whose direct solution has been subsequently used for several applications, see e.g. Hoger (1986) and Hoger (1993).

Scheidler (1994) was the first one to point out that the solution of Eq. (1) was still worth further study. He noticed in particular that the solution obtained by Hoger and Carlson (1984) for a generic $\mathbf{H}$ did not naturally yield the simpler solution previously obtained by Sidoroff (1978) and Guo (1984) for

H skew. To this end Hoger and Carlson were obliged to transform their original solution by using Rivlin's identities (Rivlin, 1955) for tensor polynomials in two variables.

For these reasons Scheidler (1994) pursued the aim of developing a more general approach which could provide the direct solution of Eq. (1) in a form specifically tailored for each kind of **H**, either generic, symmetric or skew.

Remarkably Scheidler considered also the more general tensor equation obtained from Eq. (1) by replacing the right-hand side with several isotropic functions $\mathbf{\Phi}(\mathbf{A}, \mathbf{H})$, linear in **H**, and assuming a not necessarily symmetric **A**.

The present paper is intended as a further step in the direction indicated by Scheidler. Specifically, we develop an original approach to the direct solution of the tensor Eq. (1) which appears to be simpler and more effective than the ones exploited thus far.

In particular, once some general guidelines are followed, the solution can be given a specific form from the very beginning, thus avoiding the need of devising special tensor identities to a posteriori process an already available solution.

The proposed approach is based upon the systematic use of the notion of square tensor product between second-order tensors. Such an operator, which is simply the tensor product of transformations defined in Halmos' textbook (Halmos, 1958), was first introduced in continuum mechanics by Del Piero (1979) and then used by Podio Guidugli and Virga (1987) and Guo and Podio Guidugli (1989).

By employing the notion of square tensor product, Eq. (1) can be reformulated in terms of a fourth-order tensor $\mathbb{A}$ mapping **X** to **H** so that the solution can be obtained by finding an expression for the inverse of $\mathbb{A}$.

A similar approach was partly adopted by Scheidler (1994) since he extensively used fourth-order tensors in his paper but he failed to provide explicit expressions for them and to exploit the appealing features of their algebra.

On the contrary we show that some basic properties and composition rules for the square tensor product provide the rationale for finding the solution of Eq. (1) in the case of **H** arbitrary, symmetric or skew.

Actually, the decomposition of $\mathbb{A}^{-1}$ in terms of linearly independent generators can be established either directly or by appealing to well-known representation theorems for tensor-valued isotropic functions of tensor arguments. Once $\mathbb{A}^{-1}$ has been expressed as linear combination of independent fourth-order tensors, the isotropic scalar coefficients of its representation can be obtained by straightforward algebraic manipulations.

The final form of the solution **X** can thus be decided a priori in the sense that it will depend upon the initial choice of the generators of $\mathbb{A}^{-1}$. Accordingly, if a different expression of **X** is looked for, it is more convenient to adopt a different list of generators rather than manipulating an already available solution.

Such strategy provides in addition a hint of all the possible solutions which can be obtained since $\mathbb{A}^{-1}$ is basically the linear combination of the square tensor products of $\mathbf{1}$, $\mathbf{A}$ and $\mathbf{A}^2$. Actually, the only possibilities of assuming different generators for $\mathbb{A}^{-1}$ rest in the substitution of any of these second-order tensors with the expressions resulting from the Cayley–Hamilton theorem.

In order to show the effectiveness of the proposed approach, some original solutions are provided and the solutions already available in the literature are obtained in a simpler way. The procedure for deriving additional forms of the solution of Eq. (1) is also outlined.

It is also proved that Sidoroff's approach yields the same solution as the one obtained by Hoger and Carlson (1984), a circumstance which was only claimed in their paper.

For the sake of completeness both the three- and two-dimensional cases are explicitly considered.

2. Algebraic preliminaries

Let V be a n-dimensional ($n = 2$ or $n = 3$) inner product space over the reals. We denote by Lin the space of all linear transformations (tensors) on V and by $\mathbb{L}$in the space of all tensors on Lin.

Given $\mathbf{A}, \mathbf{B} \in$ Lin, the tensor product $\mathbf{A} \otimes \mathbf{B}$ and the square tensor product $\mathbf{A} \boxtimes \mathbf{B}$ are the elements of $\mathbb{L}$in such that

$$(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B} \cdot \mathbf{C})\mathbf{A} = \mathrm{tr}(\mathbf{B}^{\mathrm{t}}\mathbf{C})\mathbf{A} \quad \forall \mathbf{C} \in \mathrm{Lin} \tag{2}$$

and

$$(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{t}} \quad \forall \mathbf{C} \in \mathrm{Lin} \tag{3}$$

While the first definition is more standard, the second one is the same as the tensor product of transformations defined by Halmos (1958). Its use in continuum mechanics dates back, to the best of the author's knowledge, to a paper by Del Piero (1979). Subsequently the square tensor product has been extensively used by Podio Guidugli and Virga (1987) and Guo and Podio Guidugli (1989).

Notice that, denoting by $\mathbf{1}$ and $\mathbb{I}$ the identity tensors in Lin and $\mathbb{L}$in respectively, the definition (3) yields

$$\mathbb{I} = \mathbf{1} \boxtimes \mathbf{1} \tag{4}$$

Further, by virtue of Eqs. (2) and (3), the following composition rules hold true

$$(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{C} \boxtimes \mathbf{D}) = (\mathbf{A}\mathbf{C}) \boxtimes (\mathbf{B}\mathbf{D}) \tag{5}$$

and

$$(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{t}}) \otimes \mathbf{D} \quad (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \boxtimes \mathbf{D}) = \mathbf{A} \otimes (\mathbf{C}^{\mathrm{t}}\mathbf{B}\mathbf{D}) \tag{6}$$

for every $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in$ Lin.

A further noteworthy property of the square tensor product is contained in the next

Proposition 2.1. The eigenvalues of $\mathbf{A} \boxtimes \mathbf{B}$ are the n^2 numbers $\lambda_a\lambda_b$ where $a, b = 1, \ldots, n$ and $\lambda_a(\lambda_b)$ are the eigenvalues of $\mathbf{A}(\mathbf{B})$. The relevant eigenvectors are given by $\mathbf{e}_a \otimes \mathbf{e}_b$ where $\mathbf{e}_a(\mathbf{e}_b)$ is the eigenvector of $\mathbf{A}(\mathbf{B})$ associated with $\lambda_a(\lambda_b)$.

Proof 2.1. By invoking Eq. (3) it follows that

$$(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{e}_a \otimes \mathbf{e}_b) = \mathbf{A}(\mathbf{e}_a \otimes \mathbf{e}_b)\mathbf{B}^{\mathrm{t}} = (\mathbf{A}\mathbf{e}_a) \otimes (\mathbf{B}\mathbf{e}_b) = \lambda_a\lambda_b(\mathbf{e}_a \otimes \mathbf{e}_b) \quad a, b = 1, \ldots, n$$

so that $\mathbf{e}_a \otimes \mathbf{e}_b$ is an eigenvector of $\mathbf{A} \boxtimes \mathbf{B}$ having $\lambda_a\lambda_b$ as associated eigenvalue.

3. Existence and uniqueness of the solution

The issue concerning the existence and the uniqueness of the solution of Eq. (1) can be addressed by invoking the result reported in Gantmacher (1977) on the solvability of the more general tensor equation in the unknown $\mathbf{X}$

$$\mathbf{AX} + \mathbf{XB} = \mathbf{H} \tag{7}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{X} \in \text{Lin}$.

We can thus state that

Proposition 3.1. The tensor Eq. (1) admits a unique solution if and only if $\mathbf{A}$ and $-\mathbf{A}$ have no eigenvalues in common.

The proof of the previous proposition is rather complicated and it is based on the properties of the Jordan normal form of a tensor. On the contrary we shall provide a simpler proof of a result, equivalent to Proposition 3.1, which holds under the additional assumption of a symmetric tensor $\mathbf{A}$ since this is the most frequent case in the applications.

Proposition 3.2. Let $\mathbf{A}$ be symmetric. The tensor Eq. (1) admits a unique solution if and only if $\mathbf{A}$ is nonsingular and it does not have two eigenvalues equal and opposite in sign.

Proof 3.2. Uniqueness of $\mathbf{X}$ is ensured if the homogeneous equation

$$\mathbf{AX} + \mathbf{XA} = \mathbf{0} \tag{8}$$

has only the trivial solution $\mathbf{X} = \mathbf{0}$.

Proof of the sufficiency is similar to the one which can be found in Gurtin's book (Gurtin, 1981) on the derivative of the square root and it is here reported to make the presentation self-contained.

Denoting by λ an eigenvalue of $\mathbf{A}$ and by $\mathbf{e}$ the relevant eigenvector it follows that

$$\mathbf{AXe} + \mathbf{XAe} = \mathbf{AXe} + \lambda\mathbf{Xe} = \mathbf{O}$$

or equivalently

$$\mathbf{AXe} = -\lambda\mathbf{Xe}$$

Hence, if $\mathbf{Xe} \neq \mathbf{O}$, $-\lambda$ is an eigenvalue of $\mathbf{A}$. Since this case is ruled out by hypothesis, we conclude that $\mathbf{Xe} = \mathbf{O}$ for every eigenvector $\mathbf{e}$ of $\mathbf{A}$. By the spectral theorem (Halmos, 1958) there is a basis for N of eigenvectors of $\mathbf{A}$ so that $\mathbf{X} = \mathbf{0}$.

Let us now turn to the *only if* part of our proposition. We proceed per absurdum.

Assume that Ker $\mathbf{A}$ is not empty and that $\mathbf{e} \in \text{Ker}\,\mathbf{A}$. Setting $\mathbf{X} = \mathbf{e} \otimes \mathbf{e}$ we should have

$$\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) + (\mathbf{e} \otimes \mathbf{e})\mathbf{A} = (\mathbf{Ae}) \otimes \mathbf{e} + \mathbf{e} \otimes (\mathbf{Ae}) = \mathbf{0}$$

since $\mathbf{Ae} = \mathbf{O}$. We thus infer that $\mathbf{A}$ must necessarily be nonsingular.

Let us now suppose that $\mathbf{A}$ has two eigenvectors, say $\mathbf{e}_2$ and $\mathbf{e}_3$, and that the associated eigenvalues are opposite in sign

$$\mathbf{A}\mathbf{e}_2 = \lambda\mathbf{e}_2; \quad \mathbf{A}\mathbf{e}_3 = -\lambda\mathbf{e}_3$$

Setting $\mathbf{X} = \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2$, Eq. (8) becomes

$$(\mathbf{A}\mathbf{e}_2) \otimes \mathbf{e}_3 + (\mathbf{A}\mathbf{e}_3) \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes (\mathbf{A}\mathbf{e}_3) + \mathbf{e}_3 \otimes (\mathbf{A}\mathbf{e}_2) = \mathbf{0}$$

once again in contrast with the hypothesis that Eq. (8) has only the trivial solution.
This completes the proof.

Remark 3.1. An alternative proof of the previous proposition hinges on the definition of the fourth-order tensor $\mathbb{A}$: Lin $\rightarrow$ Lin defined by

$$\mathbb{A} := \mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A} \tag{9}$$

which allows one to rewrite Eq. (1) in the equivalent form

$$\mathbf{AX} + \mathbf{XA} = \mathbf{H} \leftrightarrow \mathbb{A}\mathbf{X} = \mathbf{H} \tag{10}$$

We can thus state that Eq. (1) has a unique solution if and only if $\mathbb{A}$ is nonsingular or, equivalently, if all the eigenvalues of $\mathbb{A}$ are different from zero. By virtue of Proposition 2.1 this is exactly what is stated in Proposition 3.2.

The condition on the eigenvalues of $\mathbf{A}$ contained in the statement of Proposition 3.2, can be expressed more conveniently in terms of the principal invariants of $\mathbf{A}$. Actually, it can be shown, see formula (3.11) of Scheidler (1994), that

$$I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}} = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)$$

where λ_1, λ_2 and λ_3 are the eigenvalues of $\mathbf{A}$ and

$$I_{\mathbf{A}} = \mathrm{tr}\mathbf{A}, \quad II_{\mathbf{A}} = \tfrac{1}{2}[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)], \quad III_{\mathbf{A}} = \det \mathbf{A}$$

its principal invariants. We can then state

Proposition 3.3. The tensor Eq. (1) admits a unique solution if and only if

$$III_{\mathbf{A}} \neq 0 \quad \text{and} \quad I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}} \neq 0 \tag{11}$$

More elaborate results on the solvability of Eq. (1) in the case of a generic $\mathbf{A}$, even not real, can be found e.g. in Scheidler (1994).

4. Three-dimensional solutions for an arbitrary H

Let us assume that the symmetric tensor $\mathbf{A}$ fulfills the conditions reported in the Proposition 3.2. The unique solution of the tensor Eq. (1) can then be obtained by inverting the fourth-order tensor $\mathbb{A} = \mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}$. In order to find out an explicit expression for $\mathbb{A}^{-1}$ we proceed as follows.

Notice that the solution of Eq. (1) for $\mathbf{H} = \mathbf{1}$ is trivially $\mathbf{X} = \mathbf{A}^{-1}/2$. The tensor $\mathbb{A}^{-1}$ should then have a form such that $\mathbb{A}^{-1}\mathbf{1} = \mathbf{A}^{-1}/2$.

Since dim Lin = 9 a candidate expression for $\mathbb{A}^{-1}$ is

$$\mathbb{A}^{-1} = \sum_{i=1}^{9} a_i \mathbf{A}^{\beta_i} \boxtimes \mathbf{A}^{\gamma_i} \tag{12}$$

where a_i are arbitrary isotropic scalar functions of $\mathbf{A}$ such that $\sum_{i=1}^{9} a_i = 1/2$ and β_i, γ_i are arbitrary integers fulfilling the conditions $\beta_i + \gamma_i = -1$ and $\beta_i \neq \beta_j$, $\gamma_i \neq \gamma_j$ if $i \neq j$.

The tensor $\mathbb{A}^{-1}$ can however be given a simpler expression through the Cayley–Hamilton theorem according to which any tensor $\mathbf{A}$ satisfies its own characteristic equation

$$\mathbf{A}^3 - I_{\mathbf{A}}\mathbf{A}^2 + II_{\mathbf{A}}\mathbf{A} - III_{\mathbf{A}}\mathbf{1} = \mathbf{0} \tag{13}$$

Hence, by repeated recourse to the previous relation, an integer power $\mathbf{A}^k$ of the tensor $\mathbf{A}$ can be uniquely expressed as a function of the tensors $\mathbf{A}^p$, $\mathbf{A}^q$ and $\mathbf{A}^r$ where $k \neq p \neq q \neq r$.

The most natural choice is clearly $p = 0$, $q = 1$ and $r = 2$ so that we are led to assume for $\mathbb{A}^{-1}$ the following expression

$$\begin{aligned}\mathbb{A}^{-1} = {} & a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A} \boxtimes \mathbf{1}) + c(\mathbf{1} \boxtimes \mathbf{A}) + d(\mathbf{A} \boxtimes \mathbf{A}) + e(\mathbf{A}^2 \boxtimes \mathbf{1}) + f(\mathbf{1} \boxtimes \mathbf{A}^2) + g(\mathbf{A}^2 \boxtimes \mathbf{A}) + h(\mathbf{A} \boxtimes \mathbf{A}^2) \\ & + i(\mathbf{A}^2 \boxtimes \mathbf{A}^2)\end{aligned} \tag{14}$$

where $a, b, \ldots, i$ are isotropic scalar functions of $\mathbf{A}$.

Further, one expects that the symmetric structure of $\mathbb{A}$ has to entail a similar property for $\mathbb{A}^{-1}$, i.e. $b = c$, $e = f$ and $g = h$. In order to evaluate the coefficients $a, \ldots, i$ in Eq. (14) we notice that, by definition, it must be

$$\mathbb{A}^{-1}\mathbb{A} = \mathbb{A}\mathbb{A}^{-1} = \mathbb{I} = \mathbf{1} \boxtimes \mathbf{1} \tag{15}$$

or explicitly

$$\begin{aligned}& \{\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}\}\{a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A} \boxtimes \mathbf{1}) + c(\mathbf{1} \boxtimes \mathbf{A}) + d(\mathbf{A} \boxtimes \mathbf{A}) + e(\mathbf{A}^2 \boxtimes \mathbf{1}) + f(\mathbf{1} \boxtimes \mathbf{A}^2) + g(\mathbf{A}^2 \boxtimes \mathbf{A}) \\ & \quad + h(\mathbf{A} \boxtimes \mathbf{A}^2) + i(\mathbf{A}^2 \boxtimes \mathbf{A}^2)\} = \mathbf{1} \boxtimes \mathbf{1}\end{aligned}$$

Carrying out the products and recalling the composition rule (5) we obtain

$$\begin{aligned}& a(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + b(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{A} \boxtimes \mathbf{A}) + c(\mathbf{A} \boxtimes \mathbf{A} + \mathbf{1} \boxtimes \mathbf{A}^2) + d(\mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2) + e(\mathbf{A}^3 \boxtimes \mathbf{1} + \mathbf{A}^2 \boxtimes \mathbf{A}) \\ & \quad + f(\mathbf{A} \boxtimes \mathbf{A}^2 + \mathbf{1} \boxtimes \mathbf{A}^3) + g(\mathbf{A}^3 \boxtimes \mathbf{A} + \mathbf{A}^2 \boxtimes \mathbf{A}^2) + h(\mathbf{A}^2 \boxtimes \mathbf{A}^2 + \mathbf{A} \boxtimes \mathbf{A}^3) + i(\mathbf{A}^3 \boxtimes \mathbf{A}^2 + \mathbf{A}^2 \boxtimes \mathbf{A}^3) = \mathbf{1} \boxtimes \mathbf{1}\end{aligned}$$

Substituting in the previous formula the expression of $\mathbf{A}^3$ resulting from Eq. (13) and grouping together the coefficients multiplying in turn the square tensor products $\mathbb{B}_1 = \mathbf{1} \boxtimes \mathbf{1}$, $\mathbb{B}_2 = \mathbf{A} \boxtimes \mathbf{1}$, $\mathbb{B}_3 = \mathbf{1} \boxtimes \mathbf{A}$, $\mathbb{B}_4 = \mathbf{A} \boxtimes \mathbf{A}$, $\mathbb{B}_5 = \mathbf{A}^2 \boxtimes \mathbf{1}$, $\mathbb{B}_6 = \mathbf{1} \boxtimes \mathbf{A}^2$, $\mathbb{B}_7 = \mathbf{A}^2 \boxtimes \mathbf{A}$, $\mathbb{B}_8 = \mathbf{A} \boxtimes \mathbf{A}^2$, $\mathbb{B}_9 = \mathbf{A}^2 \boxtimes \mathbf{A}^2$, the condition (15) is fulfilled if and only if it turns out to be

1. $III_{\mathbf{A}} e + III_{\mathbf{A}} f = 1$

2. $a - II_{\mathbf{A}} e + III_{\mathbf{A}} h = 0$

3. $a - II_{\mathbf{A}} f + III_{\mathbf{A}} g = 0$

4. $b + c - II_{\mathbf{A}} g - II_{\mathbf{A}} h = 0$

5. $b + I_{\mathbf{A}} e + III_{\mathbf{A}} i = 0$

6. $c + I_{\mathbf{A}} f + III_{\mathbf{A}} i = 0$

7. $d + e + I_{\mathbf{A}} g - II_{\mathbf{A}} i = 0$

8. $d + f + I_{\mathbf{A}} h - II_{\mathbf{A}} i = 0$

9. $g + h + 2I_{\mathbf{A}} i = 0$ (16)

Subtracting no. 3 from no. 2 and no. 8 from no. 7 we arrive at

$$-II_{\mathbf{A}}(e - f) + III_{\mathbf{A}}(h - g) = 0$$

$$(e - f) - I_{\mathbf{A}}(h - g) = 0$$

from which we infer

$$(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}})(h - g) = 0$$

Recalling that $(I_{\mathbf{A}} II_{\mathbf{A}} - II_{\mathbf{A}}) \neq 0$, cf Eq. (11), we conclude from the previous relations that

$$h = g \quad e = f \tag{17}$$

Further, since the difference between nos. 5 and 6 of Eq. (16) yields

$$b - c + I_{\mathbf{A}}(e - f) = 0$$

it is also true that

$$b = c \tag{18}$$

as we anticipated.

By virtue of Eqs. (17) and (18), the relations (16) become

1. $2III_{\mathbf{A}} e = 1$

2. $a - II_{\mathbf{A}} e + III_{\mathbf{A}} h = 0$

4. $2b - 2II_{\mathbf{A}}h = 0$

5. $b + I_{\mathbf{A}}e + III_{\mathbf{A}}i = 0$

7. $d + e + I_{\mathbf{A}}h - II_{\mathbf{A}}i = 0$

9. $2h + 2I_{\mathbf{A}}i = 0$

We thus get from no. 1

$$e = \frac{1}{2III_{\mathbf{A}},}$$

and, from nos. 4 and 9, $b = II_{\mathbf{A}}h$ and $i = -h/I_{\mathbf{A}}$ respectively. Substituting the previous relations in no. 5 we obtain

$$h = -\frac{I_{\mathbf{A}}^2}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

so that

$$b = -\frac{I_{\mathbf{A}}^2 II_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}} \quad i = \frac{I_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

Finally, we infer from no. 2

$$a = \frac{(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})II_{\mathbf{A}} + I_{\mathbf{A}}^2 III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

and from no. 7

$$d = \frac{I_{\mathbf{A}}^3 + III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

In conclusion, setting $k = I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}}$ we can write

$$\begin{aligned} 2kIII_{\mathbf{A}}\mathbb{A}^{-1} &= [(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})II_{\mathbf{A}} + I_{\mathbf{A}}^2 III_{\mathbf{A}}](\mathbf{1} \boxtimes \mathbf{1}) - I_{\mathbf{A}}^2 II_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + (I_{\mathbf{A}}^3 + III_{\mathbf{A}})(\mathbf{A} \boxtimes \mathbf{A}) \\ &+ (I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) - I_{\mathbf{A}}^2(\mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2) + I_{\mathbf{A}}(\mathbf{A}^2 \boxtimes \mathbf{A}^2) \end{aligned}$$

so that the solution of Eq. (1) is given by

$$\begin{aligned} 2kIII_{\mathbf{A}}\mathbf{X} &= [(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})II_{\mathbf{A}} + I_{\mathbf{A}}^2 III_{\mathbf{A}}]\mathbf{H} - I_{\mathbf{A}}^2 II_{\mathbf{A}}(\mathbf{AH} + \mathbf{HA}) + (I_{\mathbf{A}}^3 + III_{\mathbf{A}})\mathbf{AHA} \\ &+ (I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})(\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2) - I_{\mathbf{A}}^2(\mathbf{A}^2\mathbf{HA} + \mathbf{AHA}^2) + I_{\mathbf{A}}(\mathbf{A}^2\mathbf{HA}^2) \end{aligned} \tag{19}$$

The previous expression coincides with the direct formula (2.3) of Hoger and Carlson (1984) and formula (5.20) of Scheidler (1994) which employed a different derivation. In addition, Eq. (19) could be deduced by specializing to the case $n = \dim V = 3$ the solution of the tensor equation $\mathbf{A}^t\mathbf{X}+\mathbf{XA}=\mathbf{H}$ obtained by Smith (1966) for an arbitrary n.

In the author's opinion the proposed approach entails some advantages with respect to the solution strategies developed in the past.

First, only simple algebraic calculations are really needed to find the solution of (1) once a representation formula for $\mathbf{X}$, or equivalently for $\mathbb{A}^{-1}$, is assigned. In this respect we recall that Sidoroff (1978) pointed out that $\mathbf{X}$ is an isotropic function of $\mathbf{A}$ and $\mathbf{H}$, linear in $\mathbf{H}$. Hence a representation formula for $\mathbf{X}$ can always be provided by resorting to well-known results on the isotropic functions of symmetric or skew-symmetric tensors, see e.g. Smith (1971), Spencer (1971), Boehler (1977) and Korsgaard (1990).

Further, by properly modifying the general expression (14) of $\mathbb{A}^{-1}$ it is possible to decide from the very beginning the structure of the solution $\mathbf{X}$ which is looked for. This can be done, for instance, by changing in Eq. (14) some, or all, the terms $\mathbf{A}^2$ with the equivalent expression as function of $\mathbf{A}^{-1}$ resulting from the Cayley–Hamilton theorem. The claimed procedure will be detailed in the next subsection.

4.1. Alternative three-dimensional solutions for an arbitrary H

It has been emphasized in the previous section that the expression (14) of $\mathbb{A}^{-1}$ is not the only possible one. To clarify this point let us re-write Eq. (14) by taking into account the symmetric structure of $\mathbb{A}^{-1}$

$$\mathbb{A}^{-1} = a_1(\mathbf{1}\boxtimes\mathbf{1}) + a_2(\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}) + a_3(\mathbf{A}\boxtimes\mathbf{A}) + a_4(\mathbf{A}^2\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}^2) + a_5(\mathbf{A}^2\boxtimes\mathbf{A}+\mathbf{A}\boxtimes\mathbf{A}^2) + a_6(\mathbf{A}^2\boxtimes\mathbf{A}^2) \tag{20}$$

Observe now that we can decide to convert all the terms $\mathbf{A}^2$ appearing in the previous formula to the equivalent ones expressed as function of $\mathbf{1}$, $\mathbf{A}$ and $\mathbf{A}^{-1}$. Actually, we derive from Eq. (13)

$$\mathbf{A}^2 = I_\mathbf{A}\mathbf{A} - II_\mathbf{A}\mathbf{1} + III_\mathbf{A}\mathbf{A}^{-1} \tag{21}$$

which yields, upon substitution in Eq. (20) and rearranging

$$\mathbb{A}^{-1} = a(\mathbf{1}\boxtimes\mathbf{1}) + b(\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}) + c(\mathbf{A}^{-1}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}^{-1}) + d(\mathbf{A}\boxtimes\mathbf{A}) + e(\mathbf{A}^{-1}\boxtimes\mathbf{A}+\mathbf{A}\boxtimes\mathbf{A}^{-1}) + f(\mathbf{A}^{-1}\boxtimes\mathbf{A}^{-1}) \tag{22}$$

We are thus led to an alternative expression of $\mathbb{A}^{-1}$ equivalent to Eq. (14).

The coefficients $a, b, \ldots, f$ in Eq. (22) can be easily found by exploiting the procedure illustrated in the previous paragraph. By imposing the condition (15) we must now require that

$$(\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A})\{a(\mathbf{1}\boxtimes\mathbf{1}) + b(\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}) + c(\mathbf{A}^{-1}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}^{-1}) + d(\mathbf{A}\boxtimes\mathbf{A}) + e(\mathbf{A}^{-1}\boxtimes\mathbf{A}+\mathbf{A}\boxtimes\mathbf{A}^{-1}) + f(\mathbf{A}^{-1}\boxtimes\mathbf{A}^{-1})\} = \mathbf{1}\boxtimes\mathbf{1}$$

Developing the products and invoking Eq. (21) we derive the following linear algebraic system in the unknowns $a, b, \ldots, f$

1. $-2II_{\mathbf{A}}b+2c=1$

2. $a+I_{\mathbf{A}}b-II_{\mathbf{A}}d+e=0$

3. $2b+2I_{\mathbf{A}}d=0$

4. $III_{\mathbf{A}}b-II_{\mathbf{A}}e+f=0$

5. $c+III_{\mathbf{A}}d+I_{\mathbf{A}}e=0$

6. $2III_{\mathbf{A}}e=0$ (23)

where each one of the previous expressions collects in turn the coefficients of the tensors $\mathbb{B}_1=\mathbf{1}\boxtimes\mathbf{1}$, $\mathbb{B}_2=\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}$, $\mathbb{B}_3=\mathbf{A}\boxtimes\mathbf{A}$, $\mathbb{B}_4=\mathbf{A}^{-1}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}^{-1}$, $\mathbb{B}_5=\mathbf{A}^{-1}\boxtimes\mathbf{A}+\mathbf{A}\boxtimes\mathbf{A}^{-1}$, $\mathbb{B}_6=\mathbf{A}^{-1}\boxtimes\mathbf{A}^{-1}$.

The solution of Eq. (23) is

$$a=\frac{I_{\mathbf{A}}^2+II_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}}-III_{\mathbf{A}})}$$

$$b=-\frac{I_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}}-III_{\mathbf{A}})}$$

$$c=-\frac{III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}}-III_{\mathbf{A}})}$$

$$d=\frac{1}{2(I_{\mathbf{A}}II_{\mathbf{A}}-III_{\mathbf{A}})}$$

$$e=0$$

$$f=\frac{I_{\mathbf{A}}III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}}-III_{\mathbf{A}})}$$

so that, having set $k=I_{\mathbf{A}}II_{\mathbf{A}}-III_{\mathbf{A}}$, we obtain

$$2k\mathbb{A}^{-1}=(I_{\mathbf{A}}^2+II_{\mathbf{A}})(\mathbf{1}\boxtimes\mathbf{1})-I_{\mathbf{A}}(\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A})-III_{\mathbf{A}}(\mathbf{A}^{-1}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}^{-1})+\mathbf{A}\boxtimes\mathbf{A}+I_{\mathbf{A}}III_{\mathbf{A}}(\mathbf{A}^{-1}\boxtimes\mathbf{A}^{-1})$$

The solution of Eq. (1) is thus given by

$$2k\mathbf{X}=(I_{\mathbf{A}}^2+II_{\mathbf{A}})\mathbf{H}-I_{\mathbf{A}}(\mathbf{AH}+\mathbf{HA})-III_{\mathbf{A}}(\mathbf{A}^{-1}\mathbf{H}+\mathbf{HA}^{-1})+\mathbf{AHA}+I_{\mathbf{A}}III_{\mathbf{A}}\mathbf{A}^{-1}\mathbf{HA}^{-1} \tag{24}$$

which coincides with formula $(5.17)_1$ of Scheidler (1994) and formula (1.15) of Ting (1996). The previous formula was also stated, without proof, by Leonov (1976) and Stickfort and Wegener (1988) for a symmetric and positive definite $\mathbf{A}$ and $\mathbf{H} \in \mathrm{Sym}$.

Using the definition of cofactor $\hat{\mathbf{A}}$ of $\mathbf{A}$ as the unique element of Lin such that

$$\hat{\mathbf{A}}\mathbf{A}^{t} = \mathbf{A}^{t}\hat{\mathbf{A}} = III_{\mathbf{A}}\mathbf{1}$$

formula $(5.18)_1$ of Scheidler (1994) and formula (1.14) of Ting (1996) can also be arrived at.

In order to show the effectiveness of the proposed approach we now derive a third expression of the solution $\mathbf{X}$ of Eq. (1). To this aim let us modify in the expression of Eq. (20) only the term $\mathbf{A}^2 \boxtimes \mathbf{A}^2$ by expressing $\mathbf{A}^2$ as function of $\mathbf{1}$, $\mathbf{A}$ and $\mathbf{A}^{-1}$ through Eq. (21).

We thus assume for $\mathbb{A}^{-1}$ the following representation formula

$$\mathbb{A}^{-1} = a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + c(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) + d(\mathbf{A} \boxtimes \mathbf{A}) + e(\mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2) + f(\mathbf{A}^{-1} \boxtimes \mathbf{A}^{-1}) \tag{25}$$

By imposing the condition (15) it turns out to be

$$(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A})\{a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + c(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) + d(\mathbf{A} \boxtimes \mathbf{A}) + e(\mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2) + f(\mathbf{A}^{-1} \boxtimes \mathbf{A}^{-1})\} = \mathbf{1} \boxtimes \mathbf{1}$$

Using Eq. (13) it is immediate to verify that the unknowns $a, b, \ldots, f$ must fulfill the linear conditions

1. $2III_{\mathbf{A}}c + 2\dfrac{II_{\mathbf{A}}}{III_{\mathbf{A}}}f = 1$

2. $a - II_{\mathbf{A}}c + III_{\mathbf{A}}e - \dfrac{I_{\mathbf{A}}}{III_{\mathbf{A}}}f = 0$

3. $2b - 2II_{\mathbf{A}}e = 0$

4. $b + I_{\mathbf{A}}c + \dfrac{1}{III_{\mathbf{A}}}f = 0$

5. $c + d + I_{\mathbf{A}}e = 0$

6. $2e = 0$ (26)

representing in turn the coefficients of the fourth-order tensors $\mathbb{B}_1 = \mathbf{1} \boxtimes \mathbf{1}$, $\mathbb{B}_2 = \mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}$, $\mathbb{B}_3 = \mathbf{A} \boxtimes \mathbf{A}$, $\mathbb{B}_4 = \mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2$, $\mathbb{B}_5 = \mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2$, $\mathbb{B}_6 = \mathbf{A}^2 \boxtimes \mathbf{A}^2$.

The solution of Eq. (26) is

$$a = \frac{I_{\mathbf{A}}^2 - II_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})}$$

$$b = 0$$

$$c = -\frac{1}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})}$$

$$d = \frac{1}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})}$$

$$e = 0$$

$$f = \frac{I_{\mathbf{A}}III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})}$$

so that

$$2k\mathbb{A}^{-1} = (I_{\mathbf{A}}^2 - II_{\mathbf{A}})(\mathbf{1} \boxtimes \mathbf{1}) - (\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) + \mathbf{A} \boxtimes \mathbf{A} + I_{\mathbf{A}}III_{\mathbf{A}}(\mathbf{A}^{-1} \boxtimes \mathbf{A}^{-1})$$

and the solution of Eq. (1) is given by

$$2k\mathbf{X} = (I_{\mathbf{A}}^2 - II_{\mathbf{A}})\mathbf{H} - (\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2) + \mathbf{A}\mathbf{H}\mathbf{A} + I_{\mathbf{A}}III_{\mathbf{A}}\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1} \tag{27}$$

It coincides with formula $(5.19)_1$ of Scheidler (1994) and it had been first derived by Jameson (1968).

It is apparent from the previous developments that further solutions of the tensor Eq. (1) can be obtained by modifying in Eq. (20), separately, each one of the pairs $(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2)$ and $(\mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2)$, as already done for $\mathbf{A}^2 \boxtimes \mathbf{A}^2$ to derive Eq. (22). Moreover we can arbitrarily modify two of the previous three pairs of square tensor products by expressing $\mathbf{A}^2$ as function of $\mathbf{1}$, $\mathbf{A}$ and $\mathbf{A}^{-1}$ through Eq. (21).

Alternatively we can even start from a different expression of Eq. (20) by assuming as generators of $\mathbf{A}^{-1}$, as an example, the square tensor products of $\mathbf{A}^{-2}$, $\mathbf{1}$ and $\mathbf{A}^2$ or the square tensor products of any triplet of integer powers of $\mathbf{A}$.

This matter shall not be pursued any further since we are not interested in making a list of the possible solutions of Eq. (1) but, rather, to illustrate the general approach which allows one to systematically generate both the solutions reported in the literature and some new ones.

For this reason we shall present in the following sections the solutions of Eq. (1) for $\mathbf{H}$ symmetric and skew.

5. Three-dimensional solution for a symmetric H

It is apparent that the expression of $\mathbf{X}$ provided by Eq. (19), or equivalently by Eqs. (24) and (27), can also be considered as direct solution of Eq. (1) for $\mathbf{H}$ symmetric. Within the framework of the proposed approach, based upon the derivation of representation formulas for $\mathbb{A}^{-1}$ this can be motivated as follows.

Denoting by Sym (Skw) the subspace of all symmetric (skew) tensors of Lin we recall that

$$\text{Lin} = \text{Sym} \oplus \text{Skw}$$

where dim Sym = 6 and dim Skw = 3.

Further, it is known that the solution **X** of Eq. (1) is symmetric if and only if $\mathbf{H} \in \text{Sym}$. Hence the representation formula for $\mathbb{A}^{-1} = (\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A})^{-1}$ is still given by Eq. (12), where the sum is now restricted only to six elements, and it must necessarily have a symmetric structure such as the ones reported in formulas (19), (24) or (27) since $\mathbb{A}^{-1}\mathbf{H} = \mathbf{X}$.

The fact that the same expression of **X** were obtained for $\mathbf{H} \in \text{Sym}$ and $\mathbf{H} \in \text{Lin}$ was also observed by Hoger and Carlson (1984) by using different arguments. Actually, they first determined the solution (19) for a symmetric **H** and they subsequently verified, by direct substitution, that the same solution was still valid for a generic **H**.

However, it has been noticed by Scheidler (1994) that no apparent simplification of the formula (19) is gained when **H** is symmetric or skew-symmetric. This is particularly disappointing for **H** skew since the solution obtained in this case by Sidoroff (1978) and Guo (1984), adopting an indirect approach, is by far simpler than the one derivable from Eq. (19). Only by using Rivlin's identities (Rivlin, 1955) for tensor polynomials in two variables were Hoger and Carlson (1984) able to convert Eq. (19) to the solution obtained by Sidoroff (1978) and Guo (1984).

This section is thus devoted to deriving direct solutions of Eq. (1) for **H** symmetric without specializing the solution (19) obtained for an arbitrary **H**. The case of **H** skew will be dealt with in the next section.

As Sidoroff (1978) first pointed out, the solution **X** of Eq. (1) is an isotropic function of **A** and **H**, linear in **H**. We can then provide a further expression of **X** by exploiting the representation theorems for tensor-valued isotropic functions of symmetric tensors (Truesdell and Noll, 1965), thus following the path first traced by Sidoroff.

Anticipating the final result we shall prove that this approach leads to the solution (2.7) which Hoger and Carlson (1984) derived by modifying Eq. (19) through Rivlin's identities (Rivlin, 1955) for tensor polynomials in two variables.

This proves the equivalence between Sidoroff's approach and the one exploited by Hoger and Carlson, a circumstance only claimed in their paper.

Let us recall that the representation theorem for isotropic tensorial functions **G** of two sym-metric tensors **D** and **E** states that **G** can be expressed as (Truesdell and Noll, 1965)

$$\mathbf{G}(\mathbf{D}, \mathbf{E}) = \psi_0\mathbf{1} + \psi_1\mathbf{D} + \psi_2\mathbf{E} + \psi_3\mathbf{D}^2 + \psi_4\mathbf{E}^2 + \psi_5(\mathbf{DE} + \mathbf{ED}) + \psi_6(\mathbf{D}^2\mathbf{E} + \mathbf{ED}^2) + \psi_7(\mathbf{E}^2\mathbf{D} + \mathbf{DE}^2)$$
$$+ \psi_8(\mathbf{D}^2\mathbf{E}^2 + \mathbf{E}^2\mathbf{D}^2)$$

where the coefficients ψ_i, $i = 0, \ldots, 8$ are isotropic scalar functions of **D** and **E**

$$\psi_i = \tilde{\psi}_i[\text{tr}\mathbf{D}, \text{tr}\mathbf{D}^2, \text{tr}\mathbf{D}^3, \text{tr}\mathbf{E}, \text{tr}\mathbf{E}^2, \text{tr}\mathbf{E}^3, \text{tr}(\mathbf{DE}), \text{tr}(\mathbf{D}^2\mathbf{E}), \text{tr}(\mathbf{DE}^2), \text{tr}(\mathbf{D}^2\mathbf{E}^2)]$$

Accordingly, since the solution **X** of Eq. (1) is an isotropic function of **A** and **H**, linear in **H**, it will admit the representation

$$\mathbf{X} = a\mathbf{H} + b(\mathbf{AH} + \mathbf{HA}) + c(\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2) + d_{00}(\mathrm{tr}\mathbf{H})\mathbf{1} + d_{01}(\mathrm{tr}\mathbf{H})\mathbf{A} + d_{02}(\mathrm{tr}\mathbf{H})\mathbf{A}^2 + d_{10}(\mathrm{tr}\mathbf{HA})\mathbf{1}$$
$$+ d_{11}(\mathrm{tr}\mathbf{HA})\mathbf{A} + d_{12}(\mathrm{tr}\mathbf{HA})\mathbf{A}^2 + d_{20}(\mathrm{tr}\mathbf{HA}^2)\mathbf{1} + d_{21}(\mathrm{tr}\mathbf{HA}^2)\mathbf{A} + d_{22}(\mathrm{tr}\mathbf{HA}^2)\mathbf{A}^2$$

The symmetry of $\mathbf{X}$ guarantees that $d_{\alpha\beta} = d_{\beta\alpha}$, $\alpha,\beta \in \{0, 1, 2\}$ so that $\mathbb{A}^{-1}$ can be given the following representation

$$\mathbb{A}^{-1} = a(\mathbf{1}\boxtimes\mathbf{1}) + b(\mathbf{A}\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}) + c(\mathbf{A}^2\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}^2) + d(\mathbf{1}\otimes\mathbf{1}) + e(\mathbf{A}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{A})$$
$$+ f(\mathbf{A}^2\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{A}^2) + g(\mathbf{A}\otimes\mathbf{A}) + h(\mathbf{A}^2\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{A}^2) + i(\mathbf{A}^2\otimes\mathbf{A}^2) \tag{28}$$

The redundancy in the number of tensors appearing in the previous formula with respect to (20) will be dealt with afterwards.

We have now to determine the coefficients $a,b,\ldots,i$ of Eq. (28) so as to enforce the condition (15) which is re-written for convenience

$$\mathbb{A}\mathbb{A}^{-1} = \mathbb{A}^{-1}\mathbb{A} = \mathbb{I} = \mathbf{1}\boxtimes\mathbf{1}$$

Recalling the composition rules, Eq. (6), we get from the first one of the previous conditions

$$a(\mathbf{A}\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}) + b[(\mathbf{A}^2\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}^2) + 2\mathbf{A}\boxtimes\mathbf{A}] + c[(\mathbf{A}^3\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}^3) + (\mathbf{A}^2\boxtimes\mathbf{A} + \mathbf{A}\boxtimes\mathbf{A}^2)] + 2d(\mathbf{A}\otimes\mathbf{1})$$
$$+ 2e(\mathbf{A}^2\otimes\mathbf{1} + \mathbf{A}\otimes\mathbf{A}) + 2f(\mathbf{A}^3\otimes\mathbf{1} + \mathbf{A}\otimes\mathbf{A}^2) + 2g(\mathbf{A}^2\otimes\mathbf{A}) + 2h(\mathbf{A}^3\otimes\mathbf{A} + \mathbf{A}^2\otimes\mathbf{A}^2) + 2i(\mathbf{A}^3\otimes\mathbf{A}^2)$$
$$= \mathbf{1}\boxtimes\mathbf{1}$$

Enforcing the condition $\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$ and combining the resulting expression with the previous one we finally obtain, by the Cayley–Hamilton Theorem, Eq. (13)

$$2III_{\mathbf{A}}c(\mathbf{1}\boxtimes\mathbf{1}) + (a - II_{\mathbf{A}}c)(\mathbf{A}\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}) + (b + I_{\mathbf{A}}c)(\mathbf{A}^2\boxtimes\mathbf{1} + \mathbf{1}\boxtimes\mathbf{A}^2) + 2b(\mathbf{A}\boxtimes\mathbf{A})$$
$$+ c(\mathbf{A}^2\boxtimes\mathbf{A} + \mathbf{A}\boxtimes\mathbf{A}^2) + 2III_{\mathbf{A}}f(\mathbf{1}\otimes\mathbf{1}) + (d - II_{\mathbf{A}}f + III_{\mathbf{A}}h)(\mathbf{A}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{A}) + (e + I_{\mathbf{A}}f + III_{\mathbf{A}}i)$$
$$\times(\mathbf{A}^2\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{A}^2) + 2(e - II_{\mathbf{A}}h)(\mathbf{A}\otimes\mathbf{A}) + (f + g + I_{\mathbf{A}}h - II_{\mathbf{A}}i)(\mathbf{A}^2\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{A}^2)$$
$$+ 2(h + I_{\mathbf{A}}i)(\mathbf{A}^2\otimes\mathbf{A}^2) = \mathbf{1}\boxtimes\mathbf{1} \tag{29}$$

It is immediate to realize that enforcement of the condition (15), represented by the previous expression, leads to an overdetermined set of linear conditions imposed on the nine unknowns $a,b,\ldots,i$.

Actually we have eleven conditions to fulfill, namely the ones concerning the five square tensor products $\mathbf{1}\boxtimes\mathbf{1}$, $(\mathbf{A}\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A})$, $(\mathbf{A}^2\boxtimes\mathbf{1}+\mathbf{1}\boxtimes\mathbf{A}^2)$, $\mathbf{A}\boxtimes\mathbf{A}$, $(\mathbf{A}^2\boxtimes\mathbf{A}+\mathbf{A}\boxtimes\mathbf{A}^2)$ plus the ones pertaining to the six tensor products $\mathbf{1}\otimes\mathbf{1}$, $(\mathbf{A}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{A})$, $(\mathbf{A}^2\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{A}^2)$, $\mathbf{A}\otimes\mathbf{A}$, $(\mathbf{A}^2\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{A}^2)$ and $\mathbf{A}^2\otimes\mathbf{A}^2$.

However Rivlin's identities (Rivlin, 1955) for tensor polynomials can be used to establish linear relations among some of the previous eleven fourth-order tensors. In particular formulas (4.25) and (4.26) of Rivlin's paper can be expressed in tensor notation as follows

$$(\mathbf{A}^2\boxtimes\mathbf{A} + \mathbf{A}\boxtimes\mathbf{A}^2) = I_{\mathbf{A}}(\mathbf{A}\boxtimes\mathbf{A}) + (\mathbf{A}^2\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{A}^2) - I_{\mathbf{A}}(\mathbf{A}\otimes\mathbf{A}) + III_{\mathbf{A}}(\mathbf{1}\otimes\mathbf{1}) - III_{\mathbf{A}}(\mathbf{1}\boxtimes\mathbf{1}) \tag{30}$$

and

$$
\begin{aligned}
(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) = & -\mathbf{A} \boxtimes \mathbf{A} + (\mathbf{A}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{A}^2) + I_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) - I_{\mathbf{A}}(\mathbf{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{A}) \\
& + \mathbf{A} \otimes \mathbf{A} + II_{\mathbf{A}}(\mathbf{1} \otimes \mathbf{1}) - II_{\mathbf{A}}(\mathbf{1} \boxtimes \mathbf{1})
\end{aligned} \tag{31}
$$

By virtue of the previous identities, Eq. (29) becomes

$$
\begin{aligned}
& [2c - II_{\mathbf{A}}(b + I_{\mathbf{A}}c) - III_{\mathbf{A}}c](\mathbf{1} \boxtimes \mathbf{1}) + [a - II_{\mathbf{A}}c + I_{\mathbf{A}}(b + I_{\mathbf{A}}c)](\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + [-(b + I_{\mathbf{A}}c) \\
& \quad + 2b + I_{\mathbf{A}}c](\mathbf{A} \boxtimes \mathbf{A}) + [II_{\mathbf{A}}(b + I_{\mathbf{A}}c) + III_{\mathbf{A}}c + 2III_{\mathbf{A}}f\,](\mathbf{1} \otimes \mathbf{1}) + [-(b + I_{\mathbf{A}}c) + d - II_{\mathbf{A}}f + III_{\mathbf{A}}h] \\
& \quad \times (\mathbf{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{A}) + (b + 2e - 2II_{\mathbf{A}}h)(\mathbf{A} \otimes \mathbf{A}) + (b + I_{\mathbf{A}}c + e + I_{\mathbf{A}}f + III_{\mathbf{A}}i)(\mathbf{A}^2 \otimes \mathbf{1} \\
& \quad + \mathbf{1} \otimes \mathbf{A}^2) + (c + f + g + I_{\mathbf{A}}h - II_{\mathbf{A}}i)(\mathbf{A}^2 \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{A}^2) + 2(h + I_{\mathbf{A}}i)(\mathbf{A}^2 \otimes \mathbf{A}^2) = \mathbf{1} \boxtimes \mathbf{1}
\end{aligned} \tag{32}
$$

from which we infer

$$a = \frac{I_{\mathbf{A}}^2 - II_{\mathbf{A}}}{I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}}}$$

$$b = 0$$

$$c = -\frac{1}{I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}}}$$

$$d = \frac{I_{\mathbf{A}}II_{\mathbf{A}}^2 + II_{\mathbf{A}}III_{\mathbf{A}} - I_{\mathbf{A}}^2 III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

$$e = -\frac{I_{\mathbf{A}}^2 II_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

$$f = \frac{I_{\mathbf{A}}II_{\mathbf{A}} + III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

$$g = \frac{I_{\mathbf{A}}^3 III_{\mathbf{A}}}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

$$h = -\frac{I_{\mathbf{A}}^2}{2(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})III_{\mathbf{A}}}$$

$$i = \frac{I_{\mathbf{A}}}{2(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}}) III_{\mathbf{A}}}$$

According to Eq. (28) and setting $k = I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}}$, the solution of Eq. (1) becomes

$$\begin{aligned} 2k III_{\mathbf{A}} \mathbf{X} = {} & 2(I_{\mathbf{A}}^2 - II_{\mathbf{A}}) III_{\mathbf{A}} \mathbf{H} - 2 III_{\mathbf{A}} (\mathbf{A}^2 \mathbf{H} + \mathbf{H} \mathbf{A}^2) + (I_{\mathbf{A}} II_{\mathbf{A}}^2 + II_{\mathbf{A}} III_{\mathbf{A}} - I_{\mathbf{A}}^2 III_{\mathbf{A}})(\mathrm{tr}\mathbf{H})\mathbf{1} \\ & - I_{\mathbf{A}}^2 II_{\mathbf{A}} [(\mathrm{tr}\mathbf{H})\mathbf{A} + (\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{1}] + (I_{\mathbf{A}} II_{\mathbf{A}} + III_{\mathbf{A}})[\mathrm{tr}\mathbf{H})\mathbf{A}^2 + (\mathrm{tr}\mathbf{H}\mathbf{A}^2)\mathbf{1}] + I_{\mathbf{A}}^3 III_{\mathbf{A}} (\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{A} \\ & - I_{\mathbf{A}}^2 [(\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{A}^2 + (\mathrm{tr}\mathbf{H}\mathbf{A}^2)\mathbf{A}] + I_{\mathbf{A}} (\mathrm{tr}\mathbf{H}\mathbf{A}^2)\mathbf{A}^2 \end{aligned} \tag{33}$$

and coincides with formula (2.7) obtained by Hoger and Carlson (1984).

Further expressions of $\mathbf{X}$ can however be obtained by combining Eq. (28) with one or both of the identities (30) and (31) and, eventually, with the identity

$$\mathbf{A}^2 \boxtimes \mathbf{A}^2 = II_{\mathbf{A}} (\mathbf{A} \boxtimes \mathbf{A}) + (\mathbf{A}^2 \otimes \mathbf{A}^2) - III_{\mathbf{A}} (\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) - II_{\mathbf{A}} (\mathbf{A} \otimes \mathbf{A}) + III_{\mathbf{A}} (\mathbf{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{A}) \tag{34}$$

stemming from the formula (4.26) of Rivlin's paper (1955).

6. Three-dimensional solutions for a skew-symmetric H

On the basis of the considerations developed in the previous section it is apparent that the solution of Eq. (1) for $\mathbf{H}$ skew can be more effectively looked for by providing a direct representation formula for $\mathbf{X}$ rather than specializing the general solution (19).

Accordingly, invoking the representation theorem for skew-symmetric tensor-valued isotropic functions of symmetric and skew tensors (Smith, 1971), we get

$$\mathbf{X} = a_1 \mathbf{H} + a_2 (\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) + a_3 (\mathbf{A}^2 \mathbf{H} + \mathbf{H}\mathbf{A}^2)$$

so that

$$\mathbb{A}^{-1} = a_1 (\mathbf{1} \boxtimes \mathbf{1}) + a_2 (\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + a_3 (\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) \tag{35}$$

We shall first derive an expression for the solution of Eq. (1) which, at the best of the author's knowledge, has not yet been reported in the literature. To this end we make use of Eq. (21) to modify Eq. (35) so as to represent $\mathbb{A}^{-1}$ in the form

$$\mathbb{A}^{-1} = a (\mathbf{1} \boxtimes \mathbf{1}) + b (\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + c (\mathbf{A}^{-1} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^{-1}) \tag{36}$$

For future reference we also need to specialize the identities (30) and (31). Observing that

$$\mathrm{tr}(\mathbf{A}^k \mathbf{H}) = 0 \quad k = 0, 1, 2$$

since $\mathbf{A}$ is symmetric and $\mathbf{H}$ is skew, we obtain

$$(\mathbf{A}^2 \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^2) = I_{\mathbf{A}} (\mathbf{A} \boxtimes \mathbf{A}) - III_{\mathbf{A}} (\mathbf{1} \boxtimes \mathbf{1}) \tag{37}$$

and

$$(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) = -(\mathbf{A} \boxtimes \mathbf{A}) + I_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) - II_{\mathbf{A}}(\mathbf{1} \boxtimes \mathbf{1}) \tag{38}$$

Finally, fulfillment of Eq. (15) starting from Eq. (36) and use of Eq. (38) provide the following relation

$$a(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + b[I_{\mathbf{A}}(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) - II_{\mathbf{A}}(\mathbf{1} \boxtimes \mathbf{1}) + \mathbf{A} \boxtimes \mathbf{A}] + c[2(\mathbf{1} \boxtimes \mathbf{1}) + (\mathbf{A}^{-1} \boxtimes \mathbf{A} + \mathbf{A} \boxtimes \mathbf{A}^{-1})]$$
$$= \mathbf{1} \boxtimes \mathbf{1} \tag{39}$$

Since the previous expression determines four linear relations among the three unknowns *a, b* and *c*, some further manipulation is needed. Namely we substitute in Eq. (39) the relation

$$\mathbf{A}^{-1} = \frac{1}{III_{\mathbf{A}}}(\mathbf{A}^2 - I_{\mathbf{A}}\mathbf{A} + II_{\mathbf{A}}\mathbf{1}) \tag{40}$$

resulting from Eq. (21) to obtain

$$\left(a + I_{\mathbf{A}}b + \frac{II_{\mathbf{A}}}{III_{\mathbf{A}}}c\right)(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + (c - II_{\mathbf{A}}b)(\mathbf{1} \boxtimes \mathbf{1}) + \left(b - \frac{I_{\mathbf{A}}}{III_{\mathbf{A}}}c\right)(\mathbf{A} \boxtimes \mathbf{A}) = \mathbf{1} \boxtimes \mathbf{1}$$

The previous relation holds if and only if

$$a = \frac{I_{\mathbf{A}}^2 + II_{\mathbf{A}}}{I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}}} \quad b = -\frac{I_{\mathbf{A}}}{I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}}} \quad c = -\frac{III_{\mathbf{A}}}{I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}}}$$

so that

$$(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})\mathbf{X} = (I_{\mathbf{A}}^2 + II_{\mathbf{A}})\mathbf{H} - I_{\mathbf{A}}(\mathbf{AH} + \mathbf{HA}) - III_{\mathbf{A}}(\mathbf{A}^{-1}\mathbf{H} + \mathbf{HA}^{-1}) \tag{41}$$

It represents an original solution of Eq. (1) for skew-symmetric **H**.

6.1. Derivation of available three-dimensional solutions for a skew-symmetric H

We now present some of the solutions already available in the literature and we briefly outline how they can be arrived at by following the approach detailed in the previous paragraphs.

First the combined use of Eqs. (37) and (38) in the expression resulting from Eq. (35) by enforcing the condition (15) provides

$$(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})\mathbf{X} = (I_{\mathbf{A}}^2 - II_{\mathbf{A}})\mathbf{H} - (\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2) \tag{42}$$

which represents the solution by Sidoroff (1978) and Guo (1984).

Further, by virtue of Eq. (38), the expression (35) of $\mathbb{A}^{-1}$ can be modified to provide the following representation

$$\mathbb{A}^{-1} = a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + c(\mathbf{A} \boxtimes \mathbf{A}) \tag{43}$$

Enforcement of Eq. (15) and use of Eq. (37) yields then

$$(I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}})\mathbf{X} = I_{\mathbf{A}}^2\mathbf{H} - I_{\mathbf{A}}(\mathbf{AH} + \mathbf{HA}) + \mathbf{AHA} \tag{44}$$

which coincides with formula $(5.1)_1$ of Scheidler (1994).

7. Two-dimensional solutions for an arbitrary H

The solution of the tensor equation

$$\mathbf{AX} + \mathbf{XA} = \mathbf{H} \tag{45}$$

for bidimensional problems can be achieved by adopting the same approach which has been illustrated in the previous sections. However the resulting expressions of $\mathbf{X}$ are quite different with respect to the corresponding three-dimensional ones so that their comprehensive presentation can be useful.

In particular we present an original solution of Eq. (45) and, for completeness, we shall also allude to the way in which further solutions reported in the literature can be obtained.

Let us first notice that the specialization of Proposition 3.2 to the present context amounts to state that Eq. (45) has a unique solution if and only if the two invariants

$$I_{\mathbf{A}} = \mathrm{tr}\mathbf{A} \quad II_{\mathbf{A}} = \det \mathbf{A}$$

are different from zero.

Following the same path of reasoning outlined at the beginning of Section 4, we are now led to assume for $\mathbb{A}^{-1}$ the following expression

$$\mathbb{A}^{-1} = a_1(\mathbf{1} \boxtimes \mathbf{1}) + a_2(\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}) + a_3(\mathbf{A} \boxtimes \mathbf{A}) \tag{46}$$

where a_1, a_2, and a_3 are isotropic scalar functions of $\mathbf{A}$.

Recalling that the Cayley–Hamilton theorem for two-dimensional tensors reads

$$\mathbf{A}^2 - I_{\mathbf{A}}\mathbf{A} + II_{\mathbf{A}}\mathbf{1} = \mathbf{0} \tag{47}$$

we can express $\mathbf{A}$ as function of $\mathbf{1}$ and $\mathbf{A}^2$ in Eq. (46) to obtain

$$\mathbb{A}^{-1} = a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A}^2 \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^2) + c(\mathbf{A}^2 \boxtimes \mathbf{A}^2) \tag{48}$$

Fulfillment of Eq. (15) yields now

$$a = \frac{I_{\mathbf{A}}^4 + II_{\mathbf{A}}^2 - I_{\mathbf{A}}^2 II_{\mathbf{A}}}{2I_{\mathbf{A}}^3 II_{\mathbf{A}}} \quad b = -\frac{I_{\mathbf{A}}^2 - II_{\mathbf{A}}}{2I_{\mathbf{A}}^3 II_{\mathbf{A}}} \quad c = \frac{1}{2I_{\mathbf{A}}^3 II_{\mathbf{A}}}$$

and hence

$$2I_{\mathbf{A}}^3 II_{\mathbf{A}}\mathbf{X} = (I_{\mathbf{A}}^4 + II_{\mathbf{A}}^2 - I_{\mathbf{A}}^2 II_{\mathbf{A}})\mathbf{H} - (I_{\mathbf{A}}^2 - II_{\mathbf{A}})(\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2) + \mathbf{A}^2\mathbf{HA}^2 \tag{49}$$

an expression which appears to be new in the literature.

A further solution of Eq. (45)

$$(2I_{\mathbf{A}} II_{\mathbf{A}})\mathbf{X} = (I_{\mathbf{A}}^2 + II_{\mathbf{A}})\mathbf{H} - I_{\mathbf{A}}(\mathbf{AH} + \mathbf{HA}) + \mathbf{AHA} \tag{50}$$

is due to Hoger and Carlson (1984). It can be obtained by enforcing Eq. (15) starting from the expression (46) for $\mathbb{A}^{-1}$ and using the Cayley–Hamilton theorem, Eq. (47), to express $\mathbf{A}^2$ as function of $\mathbf{1}$ and $\mathbf{A}$.

Finally, the solution reported by Ting (1996) in formula (6.4) of his paper

$$2I_{\mathbf{A}}\mathbf{X} = \mathbf{H} + II_{\mathbf{A}}(\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1}) \tag{51}$$

can be derived starting from the following representation of $\mathbb{A}^{-1}$

$$\mathbb{A}^{-1} = a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{A}^{-1} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A}^{-1}) + c(\mathbf{A}^{-1} \boxtimes \mathbf{A}^{-1}) \tag{52}$$

which is obtained from Eq. (46) by expressing $\mathbf{A}$ as function of $\mathbf{1}$ and $\mathbf{A}^{-1}$ through Eq. (47).

7.1. Two-dimensional solutions for H symmetric or skew

In perfect analogy with the analysis developed in Section 5, a further solution of Eq. (45) can be derived by resorting to the representation theorems for bidimensional tensor-valued isotropic functions of symmetric tensors (Korsgaard, 1990).

Actually, the solution $\mathbf{X}$ of Eq. (45) can be given the following expression

$$\mathbf{X} = a\mathbf{H} + b(\mathrm{tr}\mathbf{H})\mathbf{1} + c[(\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{1} + (\mathrm{tr}\mathbf{H})\mathbf{A}] + d(\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{A} \tag{53}$$

since it is an isotropic symmetric function of $\mathbf{A}$ and $\mathbf{H}$, linear in $\mathbf{H}$. The following representation for $\mathbb{A}^{-1}$ is thus entailed

$$\mathbb{A}^{-1} = a(\mathbf{1} \boxtimes \mathbf{1}) + b(\mathbf{1} \otimes \mathbf{1}) + c(\mathbf{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{A}) + d(\mathbf{A} \otimes \mathbf{A}) \tag{54}$$

Overdeterminacy in the set of linear conditions obtained by enforcing Eq. (15) can be dealt with by invoking Rivlin's identity for bidimensional tensor polynomials of two variables, see e.g. formula 4.7 of his paper,

$$\mathbf{A} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{A} = (\mathbf{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{A}) + I_{\mathbf{A}}(\mathbf{1} \boxtimes \mathbf{1}) - I_{\mathbf{A}}(\mathbf{1} \otimes \mathbf{1}) \tag{55}$$

The final result is

$$a = \frac{1}{I_{\mathbf{A}}} \quad b = \frac{I_{\mathbf{A}}^2 - II_{\mathbf{A}}}{2I_{\mathbf{A}}II_{\mathbf{A}}} \quad c = -\frac{1}{2II_{\mathbf{A}}} \quad d = \frac{1}{2I_{\mathbf{A}}II_{\mathbf{A}}}$$

and the relevant expression for $\mathbf{X}$

$$(2I_{\mathbf{A}}II_{\mathbf{A}})\mathbf{X} = 2II_{\mathbf{A}}\mathbf{H} + (I_{\mathbf{A}}^2 - II_{\mathbf{A}})(\mathrm{tr}\mathbf{H})\mathbf{1} - I_{\mathbf{A}}[(\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{1} + (\mathrm{tr}\mathbf{H})\mathbf{A}] + (\mathrm{tr}\mathbf{H}\mathbf{A})\mathbf{A} \tag{56}$$

coincides with the formula reported by Hoger and Carlson (1984).

The solution of Eq. (45) for $\mathbf{H}$ skew can be obtained by specializing the previous result taking into account the fact that $\mathrm{tr}\mathbf{H} = \mathrm{tr}(\mathbf{H}\mathbf{A}) = 0$. We thus get

$$I_{\mathbf{A}}\mathbf{X} = \mathbf{H}$$

a result which could be equivalently derived by invoking the representation theorem for bidimensional tensor-valued isotropic functions of skew-symmetric tensors (Korsgaard, 1990).

8. Conclusions

The analysis developed in the previous sections has allowed us to obtain new expressions for the solution of Eq. (1), both in the three- and in the two-dimensional case, as well as to provide simpler derivations of the solutions already available in the literature.

The solution of more general linear tensor equations arising in several branches of applied physics and engineering will be addressed in forthcoming papers.

Acknowledgements

The financial support of the C.N.R. (Italian National Council of Research) is gratefully acknowledged.

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